

### Hierarchy of states in the fractional quantum Hall effect

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Hierarchy of the fractional-quantum-Hall states is constructed within the framework of a recent theory. Its relationship to an earlier hierarchy and to experiment is discussed.

One of the most important objectives of any theory of the fractional-quantum-Hall effect (FQHE) is to provide an understanding of the underlying hierarchy that relates the various quantum-Hall states. Recently, a theory has been proposed by one of us<sup>1,2</sup> that provides trial wave functions for incompressible FQHE states. These trial states are identical to the Laughlin states<sup>3</sup> at filling factor  $\nu=1/m$ , where  $m$  is an odd integer, and have been generally found to be extremely accurate in few-particle calculations.<sup>4</sup> The purpose of this paper is to construct the hierarchy scheme of the FQHE states within the framework of this theory. We compare this hierarchy with the Haldane-Halperin hierarchy<sup>5</sup> and with experiment.

The hierarchy described below gives a prescription of how new incompressible states can be obtained from a given incompressible state. All incompressible states are obtained starting from the integer-quantum-Hall-effect (IQHE) states. For simplicity of exposition, we will consider only the infinite-magnetic-field ( $B$ ) limit, when all of the FQHE states (with filling factor less than 1) are spin polarized. For this we need to start with completely spin-polarized IQHE states, which are formally obtained by taking the  $g$  factor to be infinite before the limit  $B \rightarrow \infty$  is taken. The discussion of this paper can be readily extended to include the possibility of spin-unpolarized and partially spin-polarized FQHE states.

There are three elementary operations that provide new states from a given incompressible state.

(i) The first operation<sup>6</sup> is multiplication by

$$D = \prod_{j < k} (z_j - z_k)^2, \tag{1}$$

which, when applied to a state at filling factor  $\nu=p/q$ , provides a state at filling factor  $\nu'$ :

$$\chi_{\nu'} = D \chi_{\nu}, \tag{2}$$

$$\nu' \equiv D(\nu) = \frac{\nu}{2\nu+1} = \frac{p}{2p+q}. \tag{3}$$

(We will use the same notation for the operator acting on the state or acting on the filling factor; this should not lead to any confusion.) The inverse operation is defined as

$$D^{-1}(\nu) = \frac{\nu}{1-2\nu} = \frac{p}{q-2p}, \tag{4}$$

which can be applied only if  $2\nu < 1$  or  $q > 2p$ .

(ii) The second elementary operation ( $C$ ) is particle-hole conjugation<sup>7</sup> with respect to the completely occu-

ried lowest Landau level (LL):

$$\chi_{\nu'} = C \chi_{\nu}, \tag{5}$$

$$\nu' \equiv C(\nu) = 1 - \nu = \frac{q-p}{q}. \tag{6}$$

The inverse operation is given by

$$C^{-1}(\nu) = 1 - \nu = \frac{q-p}{q}. \tag{7}$$

Clearly,  $C \equiv C^{-1}$  and can be applied to  $\chi_{\nu}$  only if  $\nu < 1$ . If the wave function of the filled LL is  $\chi_1(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$ , then the particle-hole symmetric state of  $\chi_{\nu}(\mathbf{r}_1, \dots, \mathbf{r}_M)$  is explicitly given by

$$\begin{aligned} \chi_{1-\nu}(\mathbf{r}_{M+1}, \dots, \mathbf{r}_N) \\ = \int \dots \int d^2\mathbf{r}_1 \dots d^2\mathbf{r}_M \\ \times \chi_{\nu}^*(\mathbf{r}_1, \dots, \mathbf{r}_M) \chi_1(\mathbf{r}_1, \dots, \mathbf{r}_N). \end{aligned} \tag{8}$$

(iii) The final operation<sup>8</sup> that we need ( $L$ ) adds a filled LL to the state at  $\nu$  to provide a state at  $\nu'$ ,

$$\chi_{\nu'} = L \chi_{\nu}, \tag{9}$$

$$\nu' \equiv L(\nu) = 1 + \nu = \frac{p+q}{q}. \tag{10}$$

The inverse operation, which subtracts one LL, is defined as

$$L^{-1}(\nu) = \nu - 1 = \frac{p-q}{q}, \tag{11}$$

which can be used only if  $\nu > 1$ .

The particle-hole conjugation  $C$  is an exact symmetry in the limit of  $B \rightarrow \infty$ , i.e., if  $\chi_{\nu}$  is an incompressible state,  $C\chi_{\nu}$  is also incompressible.<sup>6</sup> It is a reasonable conjecture that the application of  $L$  to an incompressible state should map it to another incompressible state.<sup>8</sup> The assumption that  $D$  maps a FQHE state into another FQHE state has also been tested in several cases.<sup>6,9</sup> The most crucial and the most nontrivial ingredient in our hierarchy is that  $D$  produces an incompressible FQHE state even when it is applied to an IQHE state. It is not immediately obvious that the state of the lowest kinetic energy ends up being the state of the lowest interaction energy when multiplied by  $D$  (and projected on to the lowest LL), but it has been confirmed in extensive numerical calculations for the nontrivial cases of  $D\chi_2$  and  $D^2\chi_2$ .<sup>4</sup>

A few remarks are in order here. (i) It is clear from Eqs. (3), (6), and (10) that only odd-denominator rationals are obtained in this scheme. (ii) Any rational fraction with an odd denominator can be obtained starting from an integer. Since all integers can be obtained from 1 by repeated application of  $L$ , all of our QHE states are constructed from  $\chi_1$ . (iii) Finally, any given odd-denominator rational fraction is obtained in a unique manner. This implies that this scheme provides a unique state at each odd-denominator fraction.

In order to prove (ii) and (iii), we need to show that any rational  $\nu_0 = p_0/q_0$ , where  $p_0$  and  $q_0$  are relatively prime and  $q_0$  is odd, can be obtained from an integer by application of a string of the three (noncommuting) operators discussed above in a unique manner. Alternatively, it is sufficient to show that there is a unique sequence of the inverse operators ( $D^{-1}$ ,  $C^{-1}$ , and  $L^{-1}$ ), which, when applied to  $\nu_0$ , produces an integer. The basic idea is then to apply the operators to  $\nu_0$  in such a manner as to reduce the value of the denominator until it becomes unity. The only operator that reduces the denominator is  $D^{-1}$ , and we would therefore like to apply it again and again until we get an integer. The difficulty is that  $D^{-1}$  can be applied to a given  $\nu$  only if  $2\nu < 1$ , which is in general not the case. However, any  $\nu$  can be brought into such a form by using the other operators. Therefore we proceed as follows. If  $\nu_0 > 1$ , we apply the operator  $L^{-n}$ , where  $n$  is chosen to be the integer part of  $\nu_0$ , to obtain  $\nu'_0$  which is less than unity. If  $2\nu'_0 > 1$ , we apply the operator  $C^{-1}$  to obtain  $\nu''_0 = 1 - \nu'_0$ , which satisfies  $2\nu''_0 < 1$ . Now we apply  $D^{-1}$  to get  $\nu_1 \equiv p_1/q_1 = D^{-1}(\nu''_0)$ , where  $q_1 < q_0$ . We repeat the above steps to obtain  $\nu_2, \nu_3, \dots$  until we obtain an integer.

While multiplication by  $D$  changes the correlations substantially, there is no essential difference between the nature of correlations of the states  $\chi_\nu$ ,  $L\chi_\nu$ , and  $C\chi_\nu$ . Therefore we assume that  $D$  takes us to the next level of the hierarchy, whereas  $C$  and  $L$  keep us on the same level of the hierarchy. Thus the level at which a given state occurs in the hierarchy is equal to the number of times  $D$  is used to obtain this state from an IQHE state. It is in-

teresting to note that  $D(\nu)$  and  $\nu$  differ only in the denominator, while  $C(\nu)$ ,  $L(\nu)$ , and  $\nu$  differ only in the numerator.

The stability of a state is determined by its excitation gap. A state with a larger excitation gap is more prominent in experiments (i.e., survives to higher temperatures and is more robust to disorder) than one with a smaller excitation gap. Fortunately, intuitively plausible arguments, in conjunction with earlier calculations, allow us to say something about the relative order of stability of the various states without an actual computation of their gaps. (i)  $D\chi_\nu$  is substantially weaker than  $\chi_\nu$ . (ii) Provided they occur at the same magnetic field, the states  $\chi_\nu$  and  $C\chi_\nu$  have the same gap.<sup>7</sup> (iii) All IQHE states (which are related by  $L$ ) have the same gap ( $\hbar\omega_c$ ), provided that they occur at the same magnetic field. (iv) This is, however, not true for the FQHE states related by  $L$ ; they do not have the same gap because of the difference in the Coulomb matrix elements in different LL's. Numerical calculations<sup>10</sup> show that, due to the softer core of the Coulomb interaction, even the strongest FQHE state in the second LL, namely  $\chi_{4/3} = L\chi_{1/3}$ , is at the verge of instability. (Remember that we are assuming here that all electrons are completely spin polarized. Therefore, our 4/3 state corresponds to the experimental 7/3 state.) This suggests that application of  $L$  to a FQHE weakens it enormously, and therefore the operator  $L$  can be completely neglected to a good first approximation. This reduces the number of stable fractions drastically.

The hierarchy resulting from these simple rules has a rather complex structure. The salient features are illustrated in Figs. 1 and 2. In Fig. 1 we show the hierarchy that follows from the IQHE states by application of  $D$  and  $C$  only. In Fig. 2 we show the more complete hierarchy emanating from 1, where, again, we have neglected  $L$  beyond the second level. The experimentally observed states<sup>11</sup> are shown in thicker boxes.

Another scheme that obtains FQHE states at all odd-denominator rational fractions is the Haldane-Halperin (HH) hierarchy, which starts from the Laughlin states at  $1/m$  (Ref. 3) to build states at other fractions.<sup>5</sup> Even

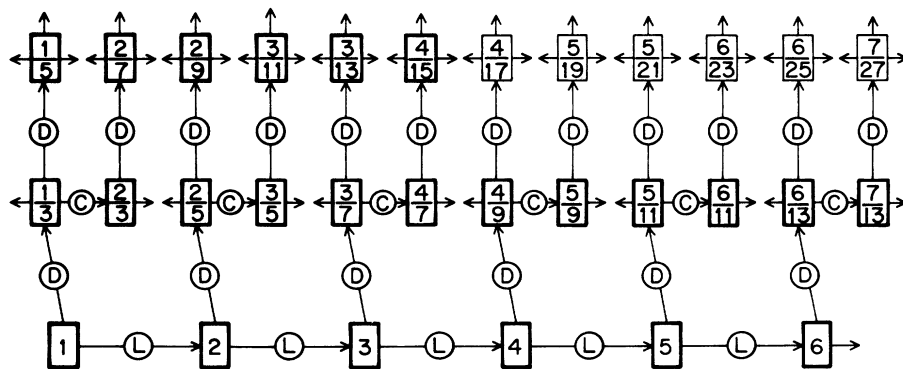


FIG. 1. Partial hierarchy generated from integers by application of the operators  $C$  and  $D$ . States at a given level of the hierarchy are shown in the same row. In this figure and Fig. 2 the states for which there is substantial observational evidence are enclosed in thick boxes. (These do not include some fractions reported in the literature that have not yet been unambiguously identified.)

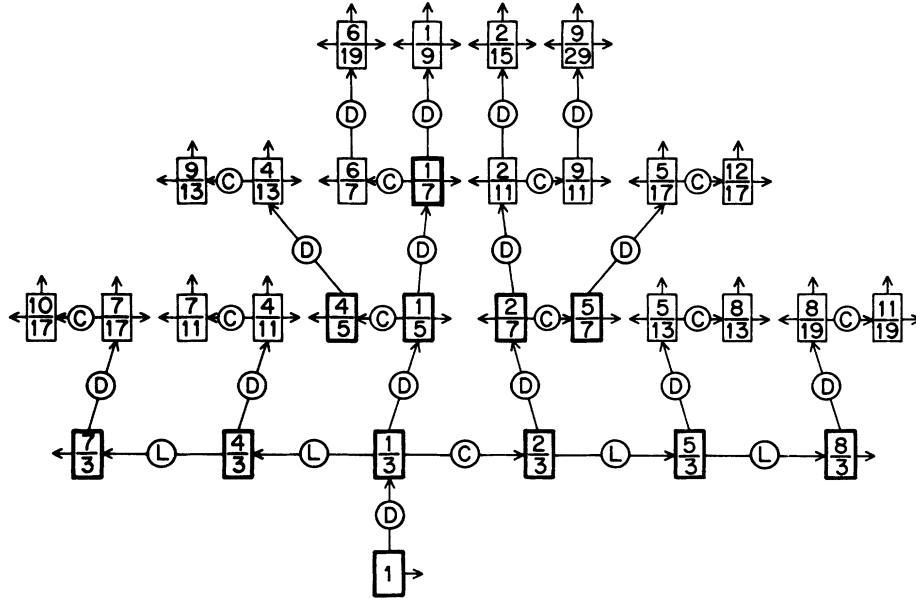


FIG. 2. More complete hierarchy emanating from  $\chi_1$ . Only the particle-hole symmetric states beyond the second level are kept for simplicity.

though both constructs produce all odd-denominator rational fractions, they have a very different hierarchy structure, and in particular, various fractions appear at different levels. For example,  $3/7$ ,  $3/11$ ,  $5/13$ , and  $5/17$  appear at the same level in the HH hierarchy, whereas in our hierarchy  $3/7$  appears at the second level,  $3/11$  and  $5/13$  appear at the third level, and  $5/17$  appears at the fourth level. As another example,  $1/3$ ,  $2/5$ ,  $3/7$ ,  $4/9$ , etc. appear at the same (second) level in our hierarchy, but at different levels in the HH hierarchy,  $n/(2n+1)$  appearing at the  $n$ th level.

Now we discuss our hierarchy in relation to experiments. As discussed earlier, application of  $C$  does not affect the stability of a state appreciably, application of  $D$  weakens a state significantly, and application of  $L$  weakens a fractional state enormously. The order of stability obtained in this manner is in good agreement with the experimental observations.<sup>11</sup> Following are some specific examples. (i)  $1/3$  ( $D\chi_1$ ),  $2/5$  ( $D\chi_2$ ),  $3/7$  ( $D\chi_3$ ), etc. are of comparable strength in experiments. (ii)  $3/7$  ( $D\chi_3$ ) is much stronger than  $3/11$  ( $DCD\chi_2$ ),  $5/13$  ( $DLCD\chi_1$ ), and  $5/17$  ( $DCDCD\chi_1$ ). (iii)  $5/13$  ( $DLCD\chi_1$ ) has not been observed even though  $6/13$  ( $D\chi_6$ ) has. (iv)  $2/7$  ( $DCD\chi_1$ ) is much weaker than  $2/5$  ( $D\chi_2$ ). (v)  $1/5$  ( $D^2\chi_1$ ) and  $2/7$  ( $DCD\chi_1$ ) are stronger than  $4/11$  ( $DLCD\chi_1$ ) and  $5/13$  ( $DLCD\chi_1$ ). In short, no spin-polarized state (with  $\nu < 1$ ) has been observed that requires the application of  $L$ , and, with the exception of  $1/7$ , no observed state requires more than two applications of  $D$ .

Another consequence of our scheme is that the transition between two FQHE states  $S\chi_n$  and  $S\chi_{n+1}$ , where  $S$  is a string of operators  $D$ ,  $C$ , and  $L$ , is closely related to

the transition between IQHE states  $\chi_n$  and  $\chi_{n+1}$ .<sup>12</sup> For example, the transition from  $\chi_{1/3} = D\chi_1$  to  $\chi_{2/5} = D\chi_2$  and the transition from  $\chi_{2/7} = DCD\chi_1$  to  $\chi_{3/11} = DCD\chi_2$  are related to the transition between  $\chi_1$  and  $\chi_2$ . This relationship between the IQHE and the FQHE has several experimentally verifiable consequences. (i) Sequences of fractions can be identified that are analogous to the IQHE sequence.<sup>13</sup> The sequences directly related to the integer sequence are precisely the prominently observed FQHE sequences. For example,  $D\chi_n$  produces the sequence  $1/3, 2/5, 3/7, \dots$ , which are the only unambiguously observed fractions in the range  $\frac{1}{2} > \nu \geq \frac{1}{3}$ . (ii) Transitions directly related to the transitions between the IQHE states (e.g.,  $1/3 \rightarrow 2/5$ ) are expected to behave qualitatively differently from those which are not (e.g.,  $1/3 \rightarrow 2/7$ ). There is experimental evidence<sup>14</sup> that this is indeed the case. (iii) It has been argued in Ref. 12 that all transitions related in this manner are expected to exhibit similar scaling behavior. This is consistent with the experiments of Engel *et al.*,<sup>15</sup> where they find that the width of the longitudinal resistance peak between  $1/3$  and  $2/5$  vanishes with the same exponent as the peak between  $1$  and  $2$ . (iv) Knowing the positions of the longitudinal resistance peak in the IQHE regime, predictions can be made about their positions in the FQHE regime.<sup>12</sup> These have also been verified experimentally.<sup>16</sup>

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- <sup>1</sup>J. K. Jain, Phys. Rev. Lett. **63**, 199 (1989).
- <sup>2</sup>J. K. Jain, Phys. Rev. B **41**, 7653 (1990); J. Phys. Chem. Solids **51**, 889 (1990).
- <sup>3</sup>R. B. Laughlin, Phys. Rev. Lett. **51**, 1395 (1983).
- <sup>4</sup>G. Dev and J. K. Jain, Phys. Rev. B **45**, 1223 (1992); E. H. Rezayi and A. H. MacDonald, *ibid.* **44**, 8395 (1991); N. Trivedi and J. K. Jain, Mod. Phys. Lett. B **5**, 503 (1991).
- <sup>5</sup>F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983); B. I. Halperin, *ibid.* **52**, 1583 (1984).
- <sup>6</sup>The importance of the operator  $D$  has been realized before. See A. H. MacDonald, G. C. Aers, and N. W. C. Dharmawardana, Phys. Rev. B **31**, 5529 (1985), for example, where they have used the operators  $C$  and  $D$  to construct states starting from 1. They thus obtain those states of our hierarchy that make no use of higher  $LL$ 's.
- <sup>7</sup>S. M. Girvin, Phys. Rev. B **29**, 6012 (1984).
- <sup>8</sup>A. H. MacDonald, Phys. Rev. B **30**, 3550 (1984).
- <sup>9</sup>See, for example, N. D'Ambrumenil and R. Morf, Phys. Rev. B **40**, 6108 (1989).
- <sup>10</sup>F. D. M. Haldane, in *The Quantum Hall Effect*, edited by R. E. Prange and S. M. Girvin (Springer-Verlag, New York, 1987).
- <sup>11</sup>D. C. Tsui, H. L. Störmer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982); H. L. Störmer *et al.*, *ibid.* **50**, 1953 (1983); A. M. Chang *et al.*, *ibid.* **53**, 997 (1984); R. G. Clark *et al.*, Surf. Sci. **170**, 141 (1986); R. B. Willet *et al.* Phys. Rev. Lett. **59**, 1776 (1987); V. J. Goldman, M. Shayegan, and D. C. Tsui, *ibid.* **61**, 881 (1988); H. W. Jiang *et al.*, *ibid.* **65**, 633 (1990).
- <sup>12</sup>J. K. Jain, S. A. Kivelson, and N. Trivedi, Phys. Rev. Lett. **64**, 1297 (1990).
- <sup>13</sup>V. J. Goldman, J. K. Jain, and M. Shayegan, Mod. Phys. Lett. B **5**, 479 (1991).
- <sup>14</sup>J. K. Wang and V. J. Goldman, Phys. Rev. Lett. **67**, 749 (1991).
- <sup>15</sup>L. Engel *et al.*, Surf. Sci. **229**, 13 (1990); H. P. Wei *et al.*, Phys. Rev. Lett. **61**, 1294 (1988).
- <sup>16</sup>V. J. Goldman, J. K. Jain, and M. Shayegan, Phys. Rev. Lett. **65**, 907 (1990).